Pismeni ispit iz Analize III, 10.02.2015.
ispit pisati isključivo hemijskom olovkom

1. Naći ekstreme funkcije $z = x + y + 4 + 4\sin x \sin y$.

2. Izračunati integral

$$\iiint\limits_{\Omega} \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

gdje je $\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq z,\, x^2 + y^2 \leq z^2\}$.

3. Izračunati krivoliniski integral prve vrste $I = \oint\limits_C \sqrt{x^2 + y^2}\,\mathrm{d}s$ gdje je C krug $x^2 + y^2 = ax$, $(a>0)$.

4. Izračunati površinu dijela lopte $x^2 + y^2 + z^2 = 3a^2$ koja se nalazi ispod parabole $x^2 + y^2 = 2az$ a iznad $x0y$ ravni.

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(#) Naći ekstreme f-je $z = x + y + 4 + 4\sin x \sin y$.

Rj.

$$\frac{\partial z}{\partial x} = 1 + 4 \cdot \cos x \sin y$$

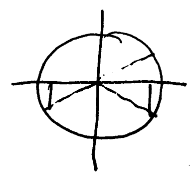
$$\frac{\partial z}{\partial y} = 1 + 4 \sin x \cos y$$

$$1 + 4\cos x \sin y = 0$$
$$1 + 4 \sin x \cos y = 0$$

$$\sin y \cos x = -\frac{1}{4} \quad (a)$$
$$\sin x \cos y = -\frac{1}{4} \quad (b)$$

(a)+(b): $\sin y \cos x + \sin x \cos y = -\frac{1}{2}$

$$\sin(y + x) = -\frac{1}{2}$$



$$y + x = \frac{7\pi}{6}$$
ili
$$y + x = \frac{11\pi}{6}$$

(a)-(b):

$$\sin y \cos x - \sin x \cos y = 0$$
$$\sin(y - x) = 0$$
$$y - x = 0 \quad \text{ili} \quad y - x = \pi$$

1° $x + y = \frac{7\pi}{6}$

$+\ -x + y = 0$

$$2y = \frac{7\pi}{6}$$
$$y = \frac{7\pi}{12} \Rightarrow x = \frac{7\pi}{12}$$

2° $x + y = \frac{7\pi}{6}$

$-x + y = \pi$

$$2y = \frac{13\pi}{6}$$
$$y = \frac{13\pi}{12} \Rightarrow x = \frac{\pi}{12}.$$

3° $x + y = \frac{11\pi}{6}$

$-x + y = 0$

$$y = \frac{11\pi}{12} \Rightarrow$$
$$x = \frac{11\pi}{12}$$

4° $y + x = \frac{11\pi}{6}$

$+\ y - x = \pi$

$$y = \frac{17\pi}{12} \Rightarrow x = \frac{5\pi}{12}$$

Stacionarne tačke su

$M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$, $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$,

$M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$, $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$$\frac{\partial^2 z}{\partial x^2} = -4 \sin x \sin y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4 \cos x \cos y$$

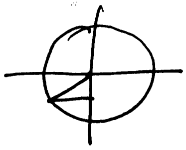
$$\frac{\partial^2 z}{\partial y^2} = -4 \sin x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad (I)$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad (II)$$

(I) + (II): $\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

(II) − (I): $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

• Za $M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$



$A = -4 \cdot \frac{1}{2}\left(\cos 0 - \cos \frac{7\pi}{6}\right) = -2\left(1 + \frac{\sqrt{3}}{2}\right) = -2 - \sqrt{3}$

$\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$B = 4 \cdot \frac{1}{2}\left(\cos \frac{7\pi}{6} + \cos 0\right) = 2\left(-\frac{\sqrt{3}}{2} + 1\right) = -\sqrt{3} + 2$

$C = -2 - \sqrt{3}$

$D = AC - B^2 = (2+\sqrt{3})^2 - (2-\sqrt{3})^2 > 0 \Rightarrow$ f-ja ima ekstrem

$A < 0$ f-ja u tački M_1 ima maksimum.

$z_{max}\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right) = \frac{7\pi}{12} + \frac{7\pi}{12} + 4 + 2 + \sqrt{3} = 6 + \sqrt{3} + \frac{7\pi}{6}$ traženi ekstrem

• Za $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$

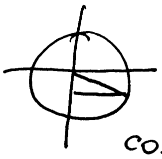
$A = -4 \cdot \frac{1}{2}\left(\cos(-\pi) - \cos \frac{7\pi}{6}\right) = -2\left(-1 + \frac{\sqrt{3}}{2}\right) = 2 - \sqrt{3}$

$B = 4 \cdot \frac{1}{2}\left(\cos \frac{7\pi}{6} + \cos(-\pi)\right) = 2\left(-\frac{\sqrt{3}}{2} - 1\right) = -\sqrt{3} - 2$

$C = 2 - \sqrt{3}$

$D = AC - B^2 = (2-\sqrt{3})^2 - (2+\sqrt{3})^2 > 0 \Rightarrow$ f-ja u tački M_2 nema ekstrema

• Za $M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$



$A = -4 \cdot \frac{1}{2}\left(\cos 0 - \cos \frac{11\pi}{6}\right) = -2\left(1 - \frac{\sqrt{3}}{2}\right) = -2 + \sqrt{3}$

$\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$B = 4 \cdot \frac{1}{2}\left(\cos \frac{11\pi}{6} + \cos 0\right) = 2\left(\frac{\sqrt{3}}{2} + 1\right) = \sqrt{3} + 2$

$C = \sqrt{3} - 2$

$D = (\sqrt{3}-2)^2 - (\sqrt{3}+2)^2 < 0 \Rightarrow$ f-ja u tački M_3 nema ekstrem

• Za $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$A = -4 \cdot \frac{1}{2}\left(\cos(-\pi) - \cos \frac{11\pi}{6}\right) = -2\left(-1 - \frac{\sqrt{3}}{2}\right) = 2 + \sqrt{3}$

$B = 4 \cdot \frac{1}{2}\left(\cos \frac{11\pi}{6} + \cos(-\pi)\right) = 2\left(\frac{\sqrt{3}}{2} - 1\right) = \sqrt{3} - 2$

$C = 2 + \sqrt{3}$, $D = AC - B^2 = (2+\sqrt{3})^2 - (\sqrt{3}-2)^2 > 0 \Rightarrow$ f-ja u tački M_4 ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right) = \frac{5\pi}{12} + \frac{17\pi}{12} + 4 + (-2 - \sqrt{3}) = 2 - \sqrt{3} + \frac{11\pi}{6}$ traženi ekstrem

(#) Izračunati integral

$$\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

gdje je $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le z,\ x^2+y^2 \le z^2\}$.

Rj.

$x^2+y^2+z^2 = z$ je jednačina sfere

$x^2+y^2 = z^2$ je jednačina čunja

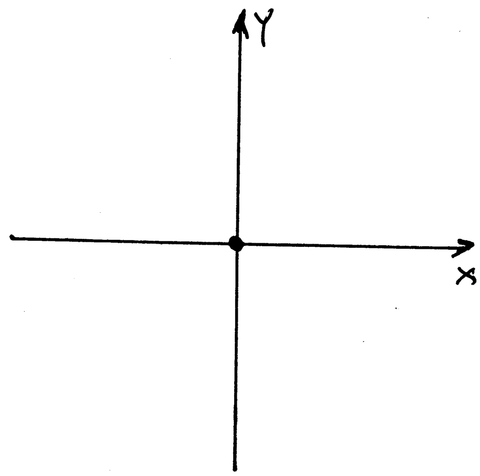
Odmah vidimo da čunj ima vrh u koordinatnom početku.

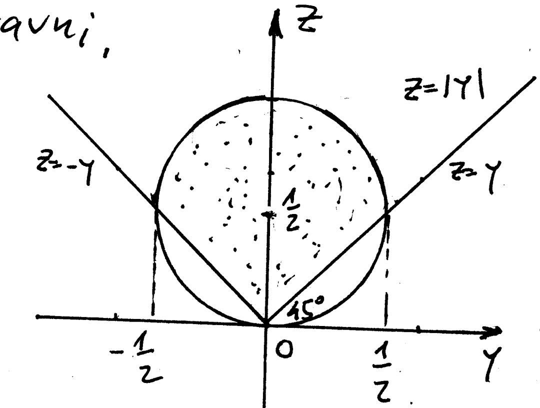
$$x^2+y^2+z^2=z$$

$$x^2+y^2+z^2-2\cdot z\cdot\frac{1}{2}+\frac{1}{4}=\frac{1}{4}$$

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$$ centar sfere je u tački $(0,0,\frac{1}{2})$ a poluprečnik $\frac{1}{2}$

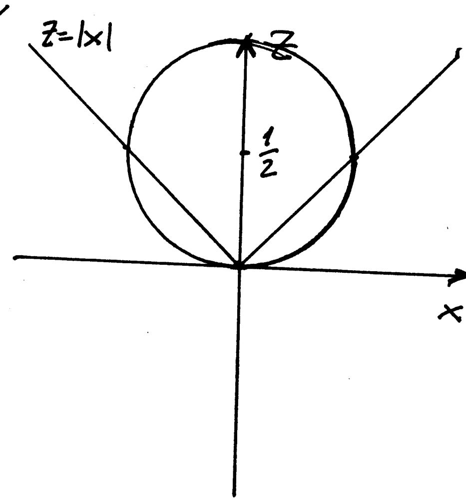
Napravimo presjeke datih figura redom sa xOy ravni, yOz ravni i yOz ravni.





SLIKA 2

yOz ravan



Sa datih slika odmah vidimo da je integral skroz teško izračunati uz pomoć pravougaonih koordinata. Uvedimo sferne koordinate.

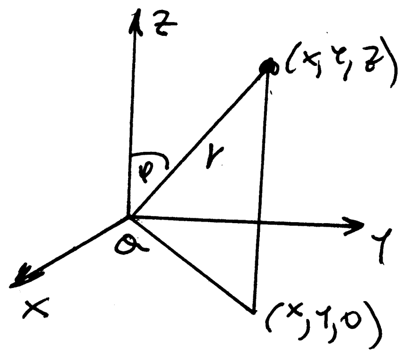
$x = r \sin\varphi \cos\theta$

$y = r \sin\varphi \sin\theta$

$z = r \cos\varphi$

$dx\,dy\,dz = r^2 \sin\varphi \, dr\, d\varphi\, d\theta$

opis točke



$\Omega \xrightarrow{\text{transformacije}} \Omega'$

Sa slike 2 čitamo granice za φ i θ. Granice za r možemo odrediti na osnovu formule

$$\Omega' : \begin{cases} 0 \leq r \leq \cos\varphi \\ 0 \leq \varphi \leq \frac{\pi}{4} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$x^2+y^2+z^2 \leq z$

tj. $r^2 \leq r\cos\varphi \quad /:r$

$r \leq \cos\varphi$.

$$\boxed{x^2+y^2+z^2 = \ldots = r^2}$$

$$\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array} \right| = \iiint\limits_{\Omega'} r^3 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^{\cos\varphi} r^3 dr \int_0^{2\pi} d\theta = 2\pi \int_0^{\pi/4} \sin\varphi\, d\varphi \int_0^{\cos\varphi} r^3 dr = 2\pi \cdot \frac{1}{4} \int_0^{\pi/4} \sin\varphi \cos^4\varphi \, d\varphi$$

$$= \left| \begin{array}{l} d(\cos\varphi) = -\sin\varphi\, d\varphi \\ \sin\varphi\, d\varphi = -\, d(\cos\varphi) \end{array} \right| = -\frac{\pi}{2} \int_0^{\pi/4} \cos^4\varphi \, d(\cos\varphi) = -\frac{\pi}{2} \cdot \frac{1}{5} \cos^5\varphi \Big|_0^{\frac{\pi}{4}} =$$

$$= -\frac{\pi}{10} \left(\left(\frac{\sqrt{2}}{2}\right)^5 - 1 \right) = \frac{\pi}{10} \left(1 - \frac{\sqrt{2}}{8} \right)$$

traženo rješenje

Izračunati krivolinijski integral prve vrste

$$I = \oint_C \sqrt{x^2+y^2}\, ds$$

gdje je C krug $x^2+y^2 = ax$ $(a>0)$.

Rj. Prisjetimo se

Ako je c kriva opisana parametarski $c: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}$ tada

$$\int_c f(x,y)\, ds = \int_{t_1}^{t_2} f(\eta(t), \mu(t)) \underbrace{\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\, dt}_{ds}$$

$$x^2+y^2 = ax$$

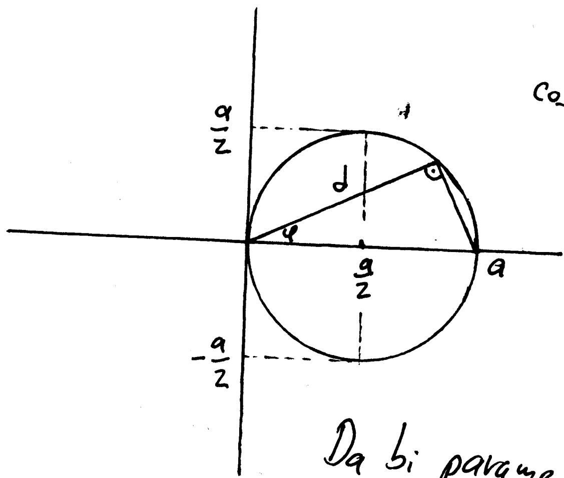
$$x^2 - ax + y^2 = 0$$

$$x^2 - 2\cdot x \cdot \frac{a}{2} + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

krug sa centrom $C(\frac{a}{2}, 0)$

poluprečnika $r = \frac{a}{2}$



$\cos\varphi = \frac{d}{a}$

$d = a\cos\varphi$

Da bi parametrizirali dati krug pomoći će nam polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

Kako r zavisi od ugla imam

$r = a\cos\varphi$

$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Parametrizacija datog kruga je

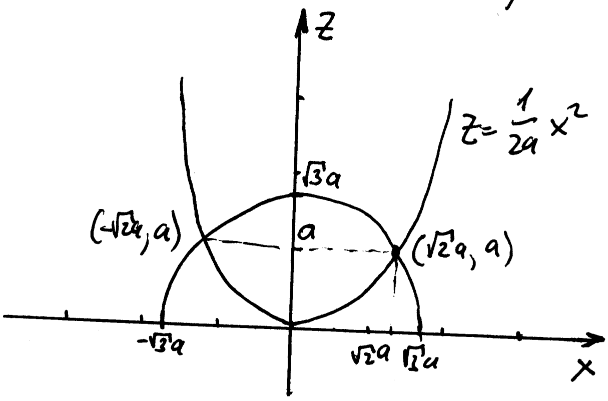
$$\left.\begin{aligned} x &= a\cos\varphi\cos\varphi = a\cos^2\varphi \\ y &= a\cos\varphi\sin\varphi = \frac{a}{2}\sin 2\varphi \end{aligned}\right\} \Rightarrow x^2+y^2 = a^2\cos^2\varphi$$

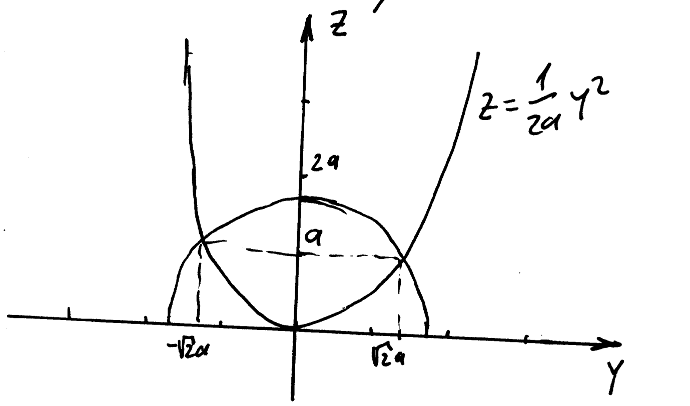
$$\left.\begin{aligned} x'_t &= 2a\cos\varphi(-\sin\varphi) = -2a\sin\varphi\cos\varphi = -a\sin 2\varphi \\ y'_t &= a\cos 2\varphi \end{aligned}\right\} \Rightarrow \sqrt{x_t'^2 + y_t'^2} = \sqrt{a^2(\sin^2 2\varphi + \cos^2 2\varphi)} = a$$

$$I = \oint_C \sqrt{x^2+y^2}\, ds = \int_{-\pi/2}^{\pi/2} a\cos\varphi \cdot a\, d\varphi = a^2 \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi = a^2 \sin\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2a^2$$ traženo rješenje

Izračunati površinu dijela lopte $x^2+y^2+z^2=3a^2$ koja se nalazi ispod parabole $x^2+y^2=2az$ a iznad xOy ravni.

Rj. Na osnovu skica presjeka datih površina sa xOz i yOz ravnima ćemo vidjeti kakva tijela su u pitanju.





$$x^2+z^2=3a^2$$
$$x^2=2az$$

$$z^2+2az-3a^2=0$$
$$D=4a^2+12a^2=16a^2$$
$$z_{1,2}=\frac{-2a\pm 4a}{2}$$
$$z_1=a \quad z_2=-3a$$

$$P=\iint_S dS$$ površinski integral prve vrste

$$z^2=3a^2-x^2-y^2$$
$$z=\pm\sqrt{3a^2-x^2-y^2}$$

U našem slučaju S je $z=\sqrt{3a^2-x^2-y^2}$ i to do one površine koji se nalazi ispod parabole

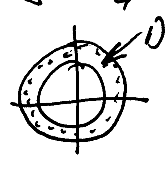
$$P=\iint_S dS=\iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z'_x=\frac{-2x}{2\sqrt{3a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{3a^2-x^2-y^2}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{3a^2-x^2-y^2}+\frac{y^2}{3a^2-x^2-y^2}=\frac{3a^2}{3a^2-x^2-y^2}$$

$$P=\sqrt{3}a\iint_D\frac{dx\,dy}{\sqrt{3a^2-x^2-y^2}}$$

gdje je D projekcija površi S na xOy ravan. U našem slučaju



Uvedimo polarne koordinate

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformiše}} D': \begin{cases} \sqrt{2}a \le r \le \sqrt{3}a \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$x^2 + y^2 = r^2$$

$$P = \sqrt{3}a \iint\limits_{D'} \frac{r\,dr\,d\varphi}{\sqrt{3a^2 - r^2}} = \sqrt{3}a \int_0^{2\pi} d\varphi \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{r\,dr}{\sqrt{3a^2 - r^2}} = \left| \begin{array}{l} 3a^2 - r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r\Big|_{\sqrt{2}a}^{\sqrt{3}a} \Rightarrow t\Big|_a^0 \end{array} \right|$$

$$= \sqrt{3}a \int_0^{2\pi} d\varphi \int_0^a \frac{t\,dt}{t} = 2a^2\sqrt{3}\pi$$

traženo rešenje

2π a